ELSEVIER Linear Algebra and its Applications 341 (2002) 119–127

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

On a question of Furuta on chaotic order

Masatoshi Fujii [a], Eizaburo Kamei [b], Ritsuo Nakamoto [c,*]

[a] *Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan*
[b] *Maebashi Institute of Technology, Kamisadori, Maebashi, Gunma 371-0816, Japan*
[c] *Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316-0033, Japan*

Received 14 December 2000; accepted 25 March 2001

Submitted by K. Okubo

Dedicated to Professor T. Ando on his 70th birthday

Abstract

The chaotic order $A \gg B$ among positive invertible operators $A, B > 0$ on a Hilbert space is introduced by $\log A \geqslant \log B$. Uchiyama's method brings us the Furuta inequality for the chaotic order from the Furuta inequality. Related to this, Furuta posed the following question: For $A, B > 0$, $A \gg B$ if and only if

$$A^{r-t} \geqslant \left\{A^{r/2}\left(A^{-t/2}B^{p}A^{-t/2}\right)^{s}A^{r/2}\right\}^{(r-t)/((p-t)s+r)} \tag{Q}$$

holds for all $p \geqslant 1$, $r \geqslant t$, $s \geqslant 1$ and $t \in [0, 1]$? Recently he gave a counterexample to the "only if" part. In this note, we point out that condition (Q) characterizes the operator order $A \geqslant B$. Moreover, (Q) characterizes the spectral order by extending the bounds of t. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Furuta inequality; Grand Furuta inequality; Chaotic order; Spectral order

1. Introduction

Throughout this note, a capital letter means a (bounded linear) operator on a Hilbert space H. An operator T is said to be *positive*, in symbol, $T \geqslant 0$ if $(Tx, x) \geqslant 0$ for all $x \in H$. In particular, we denote by $A > 0$ if $A \geqslant 0$ is invertible. The order $A \geqslant B$ for

* Corresponding author.
E-mail addresses: mfujii@cc.osaka-kyoiku.ac.jp (M. Fujii), kamei@maebashi-it.ac.jp (E. Kamei), nakamoto@base.ibaraki.ac.jp (R. Nakamoto).

PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 3 3 4 - 2

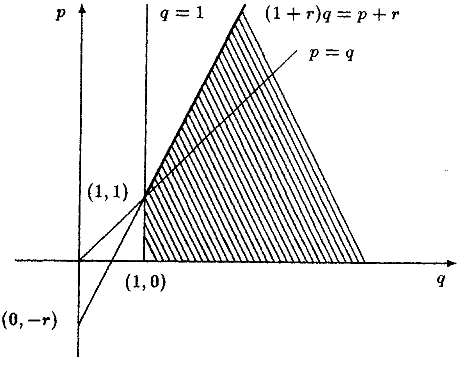

Fig. 1.

self-adjoint operators A and B is defined by $A - B \geqslant 0$. If $A \geqslant B > 0$, then the operator monotonicity of the logarithmic function yields the chaotic order $A \gg B$.

We now recall the Furuta inequality [11] which is a beautiful and historical extension of the Löwner–Heinz inequality [18,22]. We call it (FI) simply.

The Furuta inequality. *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

(i) $\left(B^{r/2}A^pB^{r/2}\right)^{1/q} \geqslant \left(B^{r/2}B^pB^{r/2}\right)^{1/q}$

and

(ii) $\left(A^{r/2}A^pA^{r/2}\right)^{1/q} \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{1/q}$

hold for $p \geqslant 0$ and $q \geqslant 1$ with

$$(1+r)q \geqslant p+r.$$

We refer [3,19] for mean theoretic proofs, and [12] for a one-page proof of it. The best possibility of the domain drawn in Fig. 1 is proved by Tanahashi [24].

On the other hand, motivated by Ando [1], the Furuta inequality for the chaotic order was shown in [4,13], cf. [6,7,28,30], which is named (FC) in the below:

If $A \gg B$ for $A, B > 0$, then

$$A^r \geqslant \left(A^{r/2}B^pA^{r/2}\right)^{r/(p+r)}$$

holds for all $p, r \geqslant 0$.

Afterwards, Furuta [14] himself generalized the Furuta inequality, which interpolates the Furuta inequality and the Ando–Hiai one [2].

The grand Furuta inequality. *If $A \geqslant B > 0$, then for each $t \in [0, 1]$,*

$$A^{1-t+r} \geqslant \left\{A^{r/2}\left(A^{-t/2}B^pA^{-t/2}\right)^s A^{r/2}\right\}^{(1-t+r)/((p-t)s+r)}$$

holds for all $s \geqslant 1,\ p \geqslant 1$ and $r \geqslant t$.

We call it (GFI) simply and refer [8] for a mean theoretic approach and [15] for a one-page proof. Tanahashi [25] also proved the best possibility of the power of (GFI) and its simplified proof is given in [10,28].

2. Uchiyama's method

Uchiyama [27] pointed out that (FI) implies (FC) by a marvelous method. In Furuta's recent paper [17], the heart of Uchiyama's method is expressed as

$$\left(1+\frac{\log X}{n}\right)^n \to X \tag{U}$$

for all $X > 0$. We now rephrase Uchiyama's proof of (FC) by using (U): Suppose that $\log A \geqslant \log B$ for $A, B > 0$ and $p, r \geqslant 0$. Then we have

$$A_n = 1+\frac{\log A}{n} \geqslant B_n = 1+\frac{\log B}{n} > 0$$

for sufficiently large n. Assuming (FI), we have

$$A_n^{1+nr} \geqslant (A_n^{nr/2} B_n^{np} A_n^{nr/2})^{(1+nr)/(np+nr)}.$$

Taking $n \to \infty$, it follows from (U) and $A_n \to 1$ that

$$A^r \geqslant (A^{r/2} B^p A^{r/2})^{r/(p+r)},$$

that is, (FC) is proved.

Next we cite a proof of (GFI) $\Rightarrow$ (FC) proved by Furuta [17], for convenience. Also we suppose that $\log A \geqslant \log B$ for $A, B > 0$ and $p, r \geqslant 0$. We apply (GFI) to $np, nr, s = 1, t = \min\{1, r\}$ and $A_n \geqslant B_n > 0$, where A_n and B_n are as in above, namely

$$A_n^{1-t+nr} \geqslant \left\{A_n^{nr/2}\left(A_n^{-t/2} B_n^{np} A_n^{-t/2}\right) A_n^{nr/2}\right\}^{(1-t+nr)/(np-t+nr)}.$$

Hence we have (FC) by taking $n \to \infty$.

3. Furuta's question

Related to (GFI) and (FC), Furuta posed the following question for himself about five years ago.

Furuta's question. For $A, B > 0$, $A \gg B$ if and only if

$$A^{r-t} \geqslant \left\{A^{r/2}\left(A^{-t/2} B^p A^{-t/2}\right)^s A^{r/2}\right\}^{(r-t)/((p-t)s+r)} \tag{Q}$$

holds for all $p \geqslant 1$, $r \geqslant t$, $s \geqslant 1$ and $t \in [0, 1]$?

Since it follows from (FC) that (Q) implies $A \gg B$ by taking $t = 0$ and $s = 1$, the converse is essential in the question. Recently Furuta himself gave a counterexample in [17]. Namely (Q) is not necessary to the chaotic order $A \gg B$. One can infer from reading between the lines that the example was based on his tough work. As a matter of fact, it was given by $A = e^X$ and $B = e^Y$, where

$$X = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}. \tag{1}$$

Then $\log A = X \geqslant Y = \log B$ and

$$A^{r-t} \not\geqslant \left\{A^{r/2}\left(A^{-t/2}B^pA^{-t/2}\right)^s A^{r/2}\right\}^{(r-t)/((p-t)s+r)} \tag{2}$$

for $r = 2$, $t = 1$, $s = 2$ and $p = 2$.

We now point out that (Q) characterizes the operator order $A \geqslant B$ for $A, B > 0$. As an immediate consequence, Furuta's question is not true because the chaotic order is exactly weaker than the operator order.

Theorem 1. *For $A, B > 0$, $A \geqslant B$ if and only if (Q) is satisfied, i.e.,*

$$A^{r-t} \geqslant \left\{A^{r/2}\left(A^{-t/2}B^pA^{-t/2}\right)^s A^{r/2}\right\}^{(r-t)/((p-t)s+r)} \tag{Q}$$

holds for all $p \geqslant 1$, $r \geqslant t$, $s \geqslant 1$ and $t \in [0, 1]$.

Proof. First of all, we recall the following Kantorovich type operator inequality, Theorem 6′ in [5], cf. also [20,29]: If $A \geqslant C$ for $A, C > 0$ and $0 < m \leqslant A \leqslant M$, then

$$\frac{(M+m)^2}{4Mm}A^2 \geqslant C^2. \tag{K}$$

We now suppose that (Q) is satisfied and $0 < m \leqslant A \leqslant M$. If we take $p = t = 1$ and $r = 2$ in (Q), then we have

$$A \geqslant \left\{A\left(A^{-1/2}BA^{-1/2}\right)^s A\right\}^{1/2}, \tag{3}$$

so that

$$\frac{(M+m)^2}{4Mm}A^2 \geqslant A\left(A^{-1/2}BA^{-1/2}\right)^s A \tag{4}$$

by (K). Hence it implies that

$$\left(\frac{(M+m)^2}{4Mm}\right)^{1/s} \geqslant A^{-1/2}BA^{-1/2} \tag{5}$$

for all $s \geqslant 1$ and so $1 \geqslant A^{-1/2}BA^{-1/2}$, or equivalently $A \geqslant B$.

Conversely, if $A \geqslant B$, then (GFI) says that for each $t \in [0, 1]$,

$$A^{1-t+r} \geqslant \left\{A^{r/2}\left(A^{-t/2}B^pA^{-t/2}\right)^s A^{r/2}\right\}^{(1-t+r)/((p-t)s+r)}$$

holds for all $s \geqslant 1$, $p \geqslant 1$ and $r \geqslant t$. Since $(r-t)/(1-t+r) \in [0,1]$, the Löwner–Heinz inequality implies that (Q) holds for all $s \geqslant 1$, $p \geqslant 1$ and $r \geqslant t$.

So the proof is complete. $\square$

Moreover, we have the following extension of Theorem 1, which says that the bounds of t is very important in such discussion:

Theorem 2. *For each $\alpha \geqslant 0$ and $A, B > 0$, $A^\alpha \geqslant B^\alpha$, where the case $\alpha = 0$ means $\log A \geqslant \log B$, if and only if* (Q) *holds for all $p \geqslant \alpha, r \geqslant t, s \geqslant 1$ and $t \in [0, \alpha]$.*

Proof. The case $\alpha = 0$ is just ensured by (FC). For each $\alpha > 0$, we can follow Theorem 1 by replacing as $A_1 = A^\alpha$ and $B_1 = B^\alpha$. $\square$

Since the self-adjoint operators do not form a complete vector lattice, Olson [23] introduced a new order among the self-adjoint operators, by which it becomes a conditionally complete lattice, cf. also [9].

Let E_t (resp. F_t) be the resolution of the identity of A (resp. B), i.e.,

$$A = \int t \, \mathrm{d}E_t \quad \text{and} \quad B = \int t \, \mathrm{d}F_t.$$

Then the spectral order $A \succeq B$ holds if $E_t \leqslant F_t$ for all t. He also proved: For positive operators A and B, $A \succeq B$ if and only if

$$A^n \geqslant B^n \quad \text{for all } n \in \mathbb{N}.$$

In addition, several useful properties of the spectral order are given by Uchiyama [26].

Anyway, as a consequence of Theorem 2, we have the following characterization of the spectral order by virtue of Olson's theorem:

Theorem 3. *For $A, B > 0$, $A \succeq B$ if and only if for each $\alpha > 0$,* (Q) *holds for all $p \geqslant \alpha, r \geqslant t, s \geqslant 1$ and $t \in [0, \alpha]$.*

In addition, we have the following slight variant of Theorem 3:

Theorem 4. *For $A, B > 0$, $A \succeq B$ if and only if* (Q) *holds for all $p, r \geqslant t \geqslant 1$ and $s \geqslant 1$.*

Proof. For the sake of completeness, we cite the proof. Suppose that (Q) holds for all $p, r \geqslant t \geqslant 1$ and $s \geqslant 1$. We take $p = t = n$ and $r = 2n$ for a given $n \in \mathbb{N}$. Then, as in the proof of Theorem 1, we have

$$A^n \geqslant \left\{ A^n \left(A^{-n/2} B^n A^{-n/2} \right)^s A^n \right\}^{1/2},$$

so that (K) implies

$$\frac{(M^n + m^n)^2}{4M^n m^n} \geqslant \left(A^{-n/2} B^n A^{-n/2}\right)^s .$$

Therefore we have $A^n \geqslant B^n$, which means that $A \succeq B$ by Olson's theorem.

The converse follows from Theorem 3. □

4. The chaotic order

As stated in front of Theorem 2, we should pay our attention to the bounds of t, cf. [20]. So we consider the case $t \leqslant 0$ and give an affirmative answer to Furuta's question in some sense.

For the sake of convenience, we cite the following useful lemma due to Furuta [14]:

Lemma F. *For $A,\ B > 0$ and $\alpha \in \mathbb{R}$,*

$$(AB^2A)^{\alpha} = AB(BA^2B)^{\alpha-1}BA,$$

or simply $(X^*X)^{\alpha} = X^*(XX^*)^{\alpha-1}X$ for invertible X.

Lemma 1. *If $A \gg B$ for $A,\ B > 0$ and $0 \leqslant p \leqslant \beta \leqslant 2p - t$ for some $t \leqslant 0$, then*

$$\left(B^{p/2} A^{-t} B^{p/2}\right)^{(\beta-p)/(p-t)} \leqslant \left(B^{p/2} A^{-u} B^{p/2}\right)^{(\beta-p)/(p-u)} \tag{6}$$

for $u \leqslant t$.

Proof. We first prove that

$$B^{p/2}\left(B^{p/2} A^{-u} B^{p/2}\right)^{(t-p)/(p-u)} B^{p/2} \leqslant A^t$$

for $p \geqslant 0$ and $u \leqslant t \leqslant 0$, cf. [21]. Actually it follows from Lemma F and (FC) that

$$\begin{aligned}
&B^{p/2}\left(B^{p/2} A^{-u} B^{p/2}\right)^{(t-p)/(p-u)} B^{p/2} \\
&\quad = \left[B^{-p/2}\left(B^{p/2} A^{-u} B^{p/2}\right)^{(p-t)/(p-u)} B^{-p/2}\right]^{-1} \\
&\quad = \left[A^{-u/2}\left(A^{-u/2} B^{p} A^{-u/2}\right)^{(u-t)/(p-u)} A^{-u/2}\right]^{-1} \\
&\quad = A^{u/2}\left(A^{-u/2} B^{p} A^{-u/2}\right)^{(t-u)/(p-u)} A^{u/2} \\
&\quad = A^{u/2}\left(A^{-u/2} B^{p} A^{-u/2}\right)^{(-u/(p-u))((t-u)/-u)} A^{u/2} \\
&\quad \leqslant A^{u/2} A^{-u(t-u)/-u} A^{u/2} \\
&\quad = A^{t}.
\end{aligned}$$

By using this, we have

$$\begin{aligned}
&\left(B^{p/2}A^{-t}B^{p/2}\right)^{(\beta-p)/(p-t)}\\
&\quad\leqslant\left(B^{p/2}B^{-p/2}\left(B^{p/2}A^{-u}B^{p/2}\right)^{(p-t)/(p-u)}\times B^{-p/2}B^{p/2}\right)^{(\beta-p)/(p-t)}\\
&\quad=\left(B^{p/2}A^{-u}B^{p/2}\right)^{(\beta-p)/(p-u)}. \qquad \square
\end{aligned}$$

Theorem 5. *For $A, B > 0$, $A \gg B$ if and only if*

$$A^{r-t} \geqslant \left\{A^{r/2}\left(A^{-t/2}B^{p}A^{-t/2}\right)^{s}A^{r/2}\right\}^{(r-t)/((p-t)s+r)} \tag{Q}$$

holds for all $p \geqslant 0$, $r \geqslant 0$, $s \in [1, 2]$ and $t \leqslant 0$.

Proof. Suppose that $A \gg B$. We put $\beta = (p-t)s + t$ and $u = t - r \leqslant t \leqslant 0$ for convenience. Then $1 \leqslant s \leqslant 2$ if and only if $p \leqslant \beta \leqslant 2p - t$ and conclusion (Q) is rephrased as follows:

$$A^{-u} \geqslant \left\{A^{(t-u)/2}\left(A^{-t/2}B^{p}A^{-t/2}\right)^{(\beta-t)/(p-t)}A^{(t-u)/2}\right\}^{-u/(\beta-u)}$$

for $0 \leqslant p \leqslant \beta \leqslant 2p - t$ and $u \leqslant t \leqslant 0$.

We now prove it, which depends on the use of the lemma (Lemma F) twice and (FC):

$$\begin{aligned}
&\left\{A^{(t-u)/2}\left(A^{-t/2}B^{p}A^{-t/2}\right)^{(\beta-t)/(p-t)}A^{(t-u)/2}\right\}^{-u/(\beta-u)}\\
&\quad=\left\{A^{-u/2}B^{p/2}\left(B^{p/2}A^{-t}B^{p/2}\right)^{(\beta-p)/(p-t)}B^{p/2}A^{-u/2}\right\}^{-u/(\beta-u)}\\
&\quad\leqslant\left\{A^{-u/2}B^{p/2}\left(B^{p/2}A^{-u}B^{p/2}\right)^{(\beta-p)/(p-u)}B^{p/2}A^{-u/2}\right\}^{-u/(\beta-u)}\\
&\quad=\left\{\left(A^{-u/2}B^{p}A^{-u/2}\right)^{(\beta-p)/(p-u)+1}\right\}^{-u/(\beta-u)}\\
&\quad=\left(A^{-u/2}B^{p}A^{-u/2}\right)^{-u/(p-u)}\\
&\quad\leqslant A^{-u}.
\end{aligned}$$

Conversely suppose that (Q) is satisfied. If we take $t = 0$ and $s = 1$, then we have (FC). So the proof is complete. $\square$

In addition, Professor Furuta kindly suggested us a related result by himself [16; Remark 2], which leads us the following parallel result to Theorem 5:

Theorem 6. *For $A, B > 0$, $A \geqslant B$ if and only if*

$$A^{1+r-t} \geqslant \left\{ A^{r/2} \left(A^{-t/2} B^p A^{-t/2} \right)^s A^{r/2} \right\}^{(1+r-t)/((p-t)s+r)} \qquad (\mathrm{Q}')$$

holds for all $p \geqslant 1$, $r \geqslant 0$, $s \in [1, 2]$ and $t \leqslant 0$.

Proof. The proof is quite similar to that of Theorem 5.

Suppose that $A \geqslant B$. We put $\beta = (p-t)s + t$ and $u = t - r \leqslant t \leqslant 0$. As in the proof of Theorem 5, $1 \leqslant s \leqslant 2$ if and only if $p \leqslant \beta \leqslant 2p - t$ and the conclusion (Q′) is rephrased as follows:

$$A^{1-u} \geqslant \left\{ A^{(t-u)/2} \left(A^{-t/2} B^p A^{-t/2} \right)^{(\beta-t)/(p-t)} A^{(t-u)/2} \right\}^{(1-u)/(\beta-u)}$$

for $1 \leqslant p \leqslant \beta \leqslant 2p - t$ and $u \leqslant t \leqslant 0$. It is proved by the use of Lemma 1, Lemma F twice and the Furuta inequality:

$$\begin{aligned}
&\left\{ A^{(t-u)/2} \left(A^{-t/2} B^p A^{-t/2} \right)^{(\beta-t)/(p-t)} A^{(t-u)/2} \right\}^{(1-u)/(\beta-u)} \\
&\quad = \left\{ A^{-u/2} B^{p/2} \left(B^{p/2} A^{-t} B^{p/2} \right)^{(\beta-p)/(p-t)} B^{p/2} A^{-u/2} \right\}^{(1-u)/(\beta-u)} \\
&\quad \leqslant \left\{ A^{-u/2} B^{p/2} \left(B^{p/2} A^{-u} B^{p/2} \right)^{(\beta-p)/(p-u)} B^{p/2} A^{-u/2} \right\}^{(1-u)/(\beta-u)} \\
&\quad = \left\{ \left(A^{-u/2} B^p A^{-u/2} \right)^{(\beta-p)/((p-u)+1)} \right\}^{(1-u)/(\beta-u)} \\
&\quad = \left(A^{-u/2} B^p A^{-u/2} \right)^{(1-u)/(p-u)} \\
&\quad \leqslant A^{1-u}.
\end{aligned}$$

Conversely suppose that (Q′) is satisfied. If we take $r = t = 0$ and $p = s = 1$, then we have $A \geqslant B$. So the proof is complete. □

References

[1] T. Ando, On some operator inequalities, Math. Ann. 279 (1987) 157–159.

[2] T. Ando, F. Hiai, Log majorization and complementary Golden–Thompson type inequalities, Linear Algebra Appl. 197/198 (1994) 113–131.

[3] M. Fujii, Furuta's inequality and its mean theoretic approach, J. Operator theory 23 (1990) 67–72.

[4] M. Fujii, T. Furuta, E. Kamei, Furuta's inequality and its application to Ando's theorem, Linear Algebra Appl. 179 (1993) 161–169.

[5] M. Fujii, S. Izumino, R. Nakamono, Y. Seo, Operator inequalities related to Cauthy–Schwarz and HÏder–McCarthy inequalities, Nihonkai Math. J. 8 (1997) 117–122.

[6] M. Fujii, J.-F. Jiang, E. Kamei, K. Tanahashi, A characterization of chaotic order and a problem, J. Inequal. Appl. 2 (1998) 149–156.
[7] M. Fujii, E. Kamei, Furuta's inequality for the chaotic order I, II, Math. Japon. 36 (1991) 603–606 and 717–722.
[8] M. Fujii, E. Kamei, Mean theoretic approach to the grand Furuta inequality, Proc. Amer. Math. Soc. 124 (1996) 2751–2756.
[9] M. Fujii, I. Kasahara, A remark on the spectral order of operators, Proc. Japan Acad. 47 (1971) 986–988.
[10] M. Fujii, A. Matsumoto, R. Nakamoto, A short proof of the best possibility for the grand Furuta inequality, J. Inequal. Appl. 4 (1999) 339–344.
[11] T. Furuta, $A \geqslant B \geqslant 0$ assures $(B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$, Proc. Amer. Math. Soc. 101 (1987) 85–88.
[12] T. Furuta, Elementary proof of an order preserving inequality, Proc. Japan Acad. 65 (1989) 126.
[13] T. Furuta, Applications of order preserving operator inequalities, Oper. Theory: Adv. Appl. 59 (1992) 180–190.
[14] T. Furuta, Extension of the Furuta inequality and Ando–Hiai log-majorization, Linear Algebra Appl. 219 (1995) 139–155.
[15] T. Furuta, Simplified proof of an order preserving operator inequality, Proc. Japan Acad. 74 (Ser. A) (1998) 114.
[16] T. Furuta, Operator functions involving order preserving inequalities, Sci. Math. 1 (1998) 141–147.
[17] T. Furuta, Results under $\log A \geqslant \log B$ can be derived from ones under $A \geqslant B \geqslant 0$ by Uchiyama's method associated with Furuta and Kantorovich type operator inequalities, Math. Inequal. Appl. 3 (2000) 423–436.
[18] E. Heinz, Beiträge zur Störungstheorie der Spectralzerlegung, Math. Ann. 123 (1951) 415–438.
[19] E. Kamei, A satellite to Furuta's inequality, Math. Japon. 33 (1988) 883–886.
[20] E. Kamei, Chaotic order and Furuta inequality, Sci. Math. Japon. 53 (2001) 289–293.
[21] E. Kamei, M. Nakamura, Remark on chaotic Furuta inequality, Sci. Math., to appear.
[22] K. Löwner, Über monotone Matrixfunktionen, Math. Z. 38 (1934) 177–216.
[23] M.P. Olson, The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice, Proc. Amer. Math. Soc. 28 (1971) 537–544.
[24] K. Tanahashi, Best possibility of the Furuta inequality, Proc. Amer. Math. Soc. 124 (1996) 141–146.
[25] K. Tanahashi, The best possibility of the grand Furuta inequality, Proc. Amer. Math. Soc. 128 (2000) 511–519.
[26] M. Uchiyama, Commutativity of selfadjoint operators, Pacific J. Math. 161 (1993) 385–392.
[27] M. Uchiyama, Some exponential operator inequalities, Math. Inequal. Appl. 2 (1999) 469–471.
[28] T. Yamazaki, Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality, Math. Inequal. Appl. 2 (1999) 473–477.
[29] T. Yamazaki, M. Yanagida, Characterizations of chaotic order associated with Kantorovich inequality, Sci. Math. 2 (1999) 37–50.
[30] M. Yanagida, Some applications of Tanahashi's result on the best possibility of Furuta inequality, Math. Inequal. Appl. 2 (1999) 297–305.